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INVESTIGATING THE GUIDANCE OFFERED TO TEACHERS IN CURRICULUM MATERIALS: THE CASE OF PROOF IN MATHEMATICS

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ABSTRACT. Despite widespread agreement that *proof* should be central to all students' mathematical experiences, many students demonstrate poor ability with it. The curriculum can play an important role in enhancing students' proof capabilities: teachers' decisions about what to implement in their classrooms, and how to implement it, are mediated through the curriculum materials they use. Yet, little research has focused on how proof is promoted in mathematics curriculum materials and, more specifically, on the guidance that curriculum materials offer to teachers to enact the proof opportunities designed in the curriculum. This paper presents an analytic approach that can be used in the examination of the guidance curriculum materials offer to teachers to implement in their classrooms the proof opportunities designed in the curriculum. Also, it presents findings obtained from application of this approach to an analysis of a popular US reform-based mathematics curriculum. Implications for curriculum design and research are discussed.

Key words: curriculum analysis, curriculum development, equivalent expressions, mathematical tasks, mathematics curriculum, proof, teachers, textbooks

INTRODUCTION

There is a growing appreciation of the importance of *proof* in school mathematics, primarily because proof is the basis of mathematical understanding and is essential for developing, establishing, and communicating mathematical knowledge. The increased emphasis on proof in many countries is reflected in both researcher and curriculum framework calls for proof to pervade students' work in mathematics throughout their schooling (Ball & Bass, 2003; Ball, Hoyles, Jahnke & Movshovitz-Hadar, 2002; Marrades & Gutiérrez, 2000; National Council of Teachers of Mathematics, 2000; Schoenfeld, 1994; Sowder & Harel, 1998; Yackel & Hanna, 2003). For example, the *Principles and Standards for School Mathematics*, an influential curriculum framework recently released in the USA by the National Council of Teachers of Mathematics (NCTM, 2000), recommends:

Instructional programs from pre-kindergarten through grade 12 should enable all students to recognize reasoning and proof as fundamental aspects of mathematics, make and

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investigate mathematical conjectures, develop and evaluate mathematical arguments and proofs, select and use various types of reasoning and methods of proof (p. 56).

The same curriculum framework also recommends that proof becomes a 'habit of mind' that is developed in different content areas, not just in geometry.

Despite increased emphasis on making proof central to all students' mathematical experiences, research shows that many students of all grade levels demonstrate poor ability with proof (e.g., Balacheff, 1988; Chazan, 1993; Healy & Hoyles, 2000; Sowder & Harel, 1998). So a critical question here is: How can we gain leverage for helping students to develop proficiency in proof? Yackel & Hanna (2003) emphasize that the most challenging undertaking for mathematics educators in their efforts to help students acquire competency in proof is "to design means to support teachers in developing forms of classroom mathematics practice that foster mathematics as reasoning and that can be carried out successfully on a large scale" (p. 234).

In this paper, I argue that one promising approach to the challenging undertaking that Yackel & Hanna (2003) describe is to equip teachers with curriculum materials (student's textbooks and teacher's editions) that design rich opportunities for students to engage in proof and that also provide teachers with the guidance necessary to enact these opportunities with their students. This approach is by itself not sufficient to yield improved mathematics instruction in the domain of proof. Research shows, for example, that teachers' beliefs (e.g., about content, curriculum materials, students, learning, teaching), experience, and knowledge mediate their use of curriculum materials (Collopy, 2003; Corey & Gamoran, 2006; Remillard, 1999, 2005; Schneider & Krajcik, 2002). Yet, curriculum materials have the potential (especially when used in conjunction with other forms of support such as professional development programs) to make a difference in students' opportunities to learn proof. Research suggests that teachers' decisions about what to implement in their classrooms, and how to implement it, are mediated through the curriculum materials they use (e.g., Beaton, Mullis, Martin, Gonzalez, Kelly & Smith, 1996; Remillard, 2000; Stein, Grover & Henningsen, 1996). Research also suggests that the basic features of the curriculum (content, organization, and sequencing) impact students' conceptions of proof (Chazan, 1993; Harel, 2001; Healy & Hoyles, 2000; Hoyles, 1997).

Despite the importance of the curriculum as a factor that can influence students' opportunities to develop proficiency in proof, little research has focused on developing and applying analytic approaches to examine the

place of proof in school mathematics curriculum programs. Application of these approaches to detailed and formative analyses of contemporary curriculum programs, supplemented by more comprehensive studies that will also include classroom-based observations and measures of student learning, will help build the necessary knowledge base for the development of curriculum materials that will support more effective instruction of proof.

In this paper, I take a step toward building a research program on the role of curriculum materials in school mathematics instruction in the domain of proof, with particular attention to a popular US reform-based mathematics curriculum and the support this curriculum provides to teachers to enact the opportunities it designs for students to engage in proofs.[1] The guidance that curriculum materials offer to teachers, in the teacher's edition, for implementing with their students the proof opportunities designed in the curriculum is important for at least three reasons: (1) teachers often have a limited understanding of proof (e.g., Knuth, 2002; Martin & Harel, 1989; Simon & Blume, 1996; Stylianides, Stylianides & Philippou, 2004; Stylianides, Stylianides & Philippou, 2007); (2) teachers often lack images of what it means to promote proof in the classroom, presumably due to the long historical tradition that has restricted proof in school mathematics to high school courses on Euclidean geometry (Stylianides (in press)); and (3) proof tasks are hard to implement in the classroom not only because they are cognitively demanding (e.g., Doyle, 1988; Stein, Grover & Henningsen, 1996) but also because students face serious difficulties with them (e.g., Chazan, 1993; Healy & Hoyles, 2000). The paper aims to address the following questions:

1. What is an analytic approach that can be used to investigate the guidance offered to teachers in curriculum materials for engaging their students in *proof tasks* (that is, curricular tasks that design opportunities for students to engage in proof)?[2]
2. What kind of guidance is provided to teachers who use a popular US reform-based mathematics curriculum for engaging their students in proof tasks? Could this guidance be further developed, and, if so, how?[3]

Sample and Analytic Approach

Sample

The sample includes all the algebra, number theory, and geometry units of the *Connected Mathematics Project* (Lappan, Fey, Fitzgerald, Friel &

Philips, 1998/2004). The *Connected Mathematics Project* (CMP) is a complete mathematics curriculum with student and teacher support materials. The guidelines set forth in the *Curriculum and Evaluation Standards* (NCTM, 1989) and the *Principles and Standards* (NCTM, 2000) have served as basis for the development of CMP. CMP's overarching goal is "to help students and teachers develop mathematical knowledge, understanding, and skill, as well as awareness and appreciation of the rich connections among mathematical strands and between mathematics and other disciplines" (Lappan, Fey, Fitzgerald, Friel & Philips, 2002, p. 1; emphasis added). CMP aims to develop students' thinking across eleven mathematical processes, including reasoning. By *reasoning* the authors mean the "[b]ringing to any problem situation the disposition and ability to observe, experiment, analyze, abstract, induce, deduce, extend, generalize, relate, and manipulate in order to find solutions or prove conjectures involving interesting and important patterns" (Lappan et al., 2002, p. 6).

My focus in this study on CMP offers two primary advantages:

1. CMP is a curriculum program that aims to promote students' opportunities to engage in proof. Given that this study intends to not only examine a phenomenon but also to develop and test an analytic approach for investigating it, it is important to choose a curriculum program that has the potential to reveal different aspects of this phenomenon. Yet, I should make clear that my analysis of CMP is not an evaluation of its efficacy in promoting students' proof capabilities. It is premature to conduct evaluation studies of curriculum programs in the domain of proof: the emphasis on proof across grade levels and content areas is recent in school mathematics (at least in the USA), and too little is known about how to best design curriculum materials that develop students' proof capabilities and support teachers in facilitating this development.
2. CMP is the most popular reform-based middle school mathematics curriculum in the USA (US Department of Education, 2000); this allows for conclusions about the curricular guidance offered to a large number of teachers to engage their students in proof.

I focused my analysis on the algebra, number theory, and geometry units of CMP mainly because these content areas are important in the middle grades and are likely to concentrate the bulk of the curricular opportunities designed for students to engage in proof. To identify the CMP segments associated with these three content areas, I considered the

major content area each CMP unit is promoting (as specified by the curriculum authors). CMP has a total of 24 units, eight in each grade level; the sample included 12 of these units.

Analytic Approach

Figure 1 summarizes the process I followed in the analysis. The analysis involved: (1) deciding whether each task in the sample designed an opportunity for students to engage in proof (called *proof task*) or not (called *non-proof task*); and (2) coding each proof task according to whether the teacher's edition offered only possible solution(s) to the task (*solution only* category) or whether it offered additionally to the teacher other forms of guidance related to the task (*solution with additional guidance* category). Next, I present my definition of proof, explain how I distinguished between proof tasks and non-proof tasks, elaborate on the two categories about the guidance offered to the teacher, and specify the unit of analysis in the study.

Defining Proof

A *proof* is a valid argument based on accepted truths for or against a mathematical claim. The term 'argument' denotes a connected sequence of assertions. The term 'valid' indicates that these assertions are connected by means of accepted canons of correct inference such as modus ponens and modus tollens. The term 'valid' should be understood in the context of what is typically agreed upon in the field of mathematics nowadays. Of course, this is not to say that this term has universal meaning in the field of mathematics nowadays, but it is beyond the scope of this paper to elaborate on this issue. The term 'accepted truths' includes the axioms, theorems, definitions, and modes of reasoning that a particular community may take as shared at a given time. The 'community' is considered to be the hypothetical classroom that covers serially all the parts of CMP and in the ways designed by the curriculum

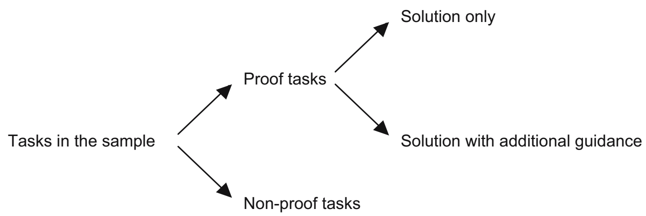


Figure 1. Process followed in the curriculum analysis

authors. Valid arguments by counterexample, contradiction, mathematical induction, contraposition, and exhaustion are examples of proofs. I do not associate proofs with any particular representational form such as the use of algebra. For example, the following argument for the claim '$odd + odd = even$' would qualify as a proof: 'All odd numbers if you circle them by twos there's one left over. So, if you add two odd numbers, the two ones left over from the two odd numbers will group together and will make an even number. This is because all even numbers if you circle them by twos there's none left over.' (adapted from Ball & Bass, (2003))

Distinguishing Between Proof Tasks and Non-proof Tasks

An important issue in the analysis was how to distinguish between proof tasks and non-proof tasks, especially in light of the fact that the *actual formulation* of tasks-that is, how the tasks played out in classroom practice-is not available when one analyzes curriculum materials. To code the tasks, I developed a way to make reasonable inferences about the *expected formulation* of the tasks-that is, the path students are anticipated (by the curriculum authors) to follow in solving the tasks. I determined the expected formulation of the tasks by working them out in the order they appear in the curriculum and by considering together the following three factors: (1) the approach suggested by the student's textbook; (2) the approach suggested in the teacher's edition; and (3) the student's expected level of knowledge and understanding when encountering a certain task. I determined the latter by looking at what preceded the given task in the student's textbook and considering it to be known to the students (e.g., What theorems, definitions, mathematical conventions, and methods were the students expected to know up to that point?). In short, the analysis of the textbooks to decide whether a task was a proof task or a non-proof task was a hypothetical enterprise: What opportunities to engage in proofs *would* students have if their classes covered serially all the parts in their curriculum materials and in the ways designed by the curriculum authors? This approach to the curriculum analysis does not suggest that teachers should work through a particular set of curriculum materials religiously. In other words, this approach to analyzing curriculum materials does not imply a certain perspective on how teachers should implement those curriculum materials in their classrooms. Rather, this approach offers a meaningful and reliable way to draw inferences about the expected formulation of the tasks when these tasks are implemented in a way that is consistent with the curriculum authors' intentions.

Finally, I tested the inter-rater agreement of the coding system by comparing my codes with the codes of a second rater in a purposeful sub-sample, which included (based on my codes) an approximately equal number of proof tasks and non-proof tasks. The reliability value was based on our decisions about whether or not each task in the sub-sample was a proof task or a non-proof task: 90.2%, kappa statistic=0.8 (Siegel & Castellan, 1988).

Analyzing the Guidance Offered to Teachers

Each proof task in the sample was coded into the *solution with additional guidance* or the *solution only* categories, according to whether or not the guidance offered to teachers in the teacher's edition was going beyond possible solution(s) to the task. For this part of my analysis, I considered only the teacher's edition, which contains the student activities and sidebar notes to teachers regarding how to implement those activities. I did not consider other support materials for teachers (e.g., assessment resources, CD-ROM activities), because research suggests that, under the busy conditions of their everyday practice, teachers tend to consult only materials that appear in the same books and are in close proximity to their daily lessons (Stein & Kim, 2006).

Proof tasks in the 'solution with additional guidance' category included at least one characteristic from those listed in the three (interrelated) *forms of additional guidance* (hereafter referred to as *FAG*) that I describe below. I developed these FAG based on research work (especially Ball & Cohen, 1996; Davis & Krajcik, 2005) that suggests possible ways in which K-12 curriculum materials can promote teacher learning (not necessarily in the particular domain of proof). Also, in developing these three FAG, I paid special attention to the fact that proof is a hard-to-teach and hard-to-learn topic (for both students and teachers).

FAG 1: *Explanations about why students' engagement in a proof task matters* (drawing on the work of Ball & Cohen, 1996; Davis & Krajcik, 2005; Remillard, 2000, 2005; Stein & Kim, 2006). Curriculum materials can make visible to teachers the goals they may try to accomplish as they implement proof tasks in their classrooms. This form of guidance is important because proof tasks are likely to cause difficulties for students (and possibly frustration) and, thus, teachers need to know the mathematical and pedagogical reasons for engaging their students in these tasks.

FAG 2: *Cautious points on how to manage student approaches to a proof task* (drawing on the work of Ball & Cohen, 1996; Collopy, 2003; Davis & Krajcik, 2005; Remillard, 2000; Stein & Kim, 2006). Curriculum materials can help teachers anticipate student approaches (both correct and incorrect) to a proof task and give suggestions for how to deal with them. For example, curriculum materials can use research findings on students' understanding of proof to specify typical misconceptions that are likely to surface as students engage in a proof task, and suggest ways with which teachers can manage those misconceptions. By so doing, curriculum materials can reinforce teachers' pedagogical content knowledge in the domain of proof (Shulman, 1986).

FAG 3: *Discussions that support teachers' content knowledge of proof* (drawing on the work of Ball & Cohen, 1996; Davis & Krajcik, 2005; Wang & Paine, 2003). Curriculum materials can include discussions to help teachers develop a thorough understanding of the mathematical ideas embedded in particular proof tasks so that teachers can have a good sense of the mathematical terrain covered by each task and be better prepared to implement it in their classrooms. These discussions can focus on issues of proof that research suggests that teachers have difficulties with.

The three FAG would be equally applicable when examining the guidance offered to teachers to implement curricular tasks related to other hard-to-teach and hard-to-learn topics of the school curriculum such as *problem solving* in mathematics and *communicating and justifying findings* in science. I revisit this issue at the last section of the paper.

Defining the Unit of Analysis

Different curriculum programs follow different marking systems for their activities (e.g., exercises, problems). For example, in the USA, reform-based mathematics curriculum programs mark as one activity what conventional curriculum programs would typically mark as many different ones. Therefore, it was important to define the unit of analysis in the study so that the same unit of analysis could be applied in analyses of curriculum programs that follow different marking systems. This would allow comparisons between the results of this study and future studies that would use the same approach to investigate other curriculum programs.

I decided to consider the task as the unit of analysis in this study, where by *task* I mean any activity or parts of it that has a separate marker in the student's textbook. This conceptualization of task has been developed to serve the purposes of curriculum analyses as explained and it should not be interpreted as a reaction to other conceptualizations used by researchers to serve different purposes (e.g., Stein et al., 1996).

Results and Discussion

I examined a total of 4,855 tasks. For 277 tasks, there was not enough information in the curriculum materials to code them as proof tasks or non-proof tasks. Therefore, I eliminated these 277 tasks from the analysis. About 5% of the remaining 4,578 tasks in the sample (227 tasks) were coded as proof tasks. In 90% of the proof tasks the guidance offered to teachers was limited to possible solution(s) to the task (solution only); in the remaining 10% of the proof tasks there were additional forms of guidance to the teacher (solution with additional guidance).

The rest of this section is organized into two parts, which complement each other in presenting my analysis of the guidance provided to teachers for engaging their students in proof tasks. The first part is a discussion of the guidance offered to teachers in individual proof tasks drawn from the sample. These are purposefully selected examples, illustrative of the two coding categories: 'solution only' and 'solution with additional guidance.' I use as a lens the three FAG elaborated earlier (especially ideas for teacher guidance that consider students' and teachers' difficulties with proof; cf. FAG 2 and FAG 3, respectively) to suggest possible ways in which the guidance offered to teachers in tasks that were coded in the 'solution only' category could be further developed to qualify for the 'solution with additional guidance' category.

The second part is a discussion of the guidance offered to teachers in a set of problems that belong to the topic of *equivalence of symbolic expressions*. I use again as a lens for my discussion of these problems the three FAG elaborated earlier, especially FAG 2 and FAG 3. I do not claim that the guidance offered to teachers in this set of problems is representative of the guidance provided to teachers in CMP. The reason for which I selected this set of problems is that it offers me the opportunity to discuss issues of curriculum design that go beyond CMP and the particular domain of proof, and that would be difficult to discuss in the context of individual unrelated tasks. Specifically, this set of

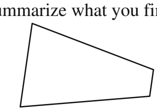
problems raises important issues related to: (1) the use of language in mathematics curriculum materials; (2) the degree of coherence among the different opportunities designed in the curriculum; and (3) the challenges that curriculum authors face in regard to the amount of, and the best way to present the, guidance to teachers in a teacher's edition.

Individual Tasks Illustrative of the 'Solution Only' and 'Solution with Additional Guidance' Gategories

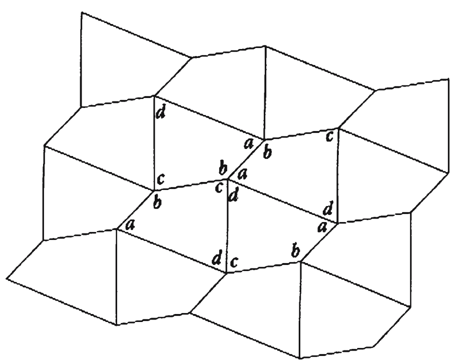
The problem in Figure 2 (example 1) is a proof task that was coded as providing 'solution with additional guidance' because teachers are not only offered a way to solve the task, but are also cautioned about possible difficulties that students can face as they try to solve the task (e.g., as

Example 1 (from CMP unit, *Shapes and Designs*)

Problem (p. 13): An *irregular polygon*, such as the one shown below, is a polygon in which the sides are not all the same length. Choose an irregular quadrilateral…. Cut out several copies of your figure and see whether you can use them to tile a surface. Summarize what you find about using quadrilaterals to tile a surface.



Commentary in the Teacher's Edition about this Problem (p. 14d): An irregular quadrilateral will tile a surface. Because there is a total of 360º in the angles of quadrilaterals, one of each angle must surround each vertex point in the pattern. As with a triangle, sides that are the same length must be put together. A way to help students see the pattern and to see that the pattern works for any quadrilateral is to label the sides. This shows that there are copies of each of the angles of the quadrilateral around each vertex point, and that the sides are always placed so that the lengths match and the labels are reversed.



For the Teacher: Tiling with an Irregular Quadrilateral

It will not be easy for students to see how to tile using the irregular quadrilateral. If no one discovers how to do a tiling with that shape (or for that matter, any one of the other shapes they are exploring), we recommend that you leave it as an open question for students to work on (rather than showing them yourself).

Figure 2. Guidance offered to teachers in proof tasks: example 1

students try to observe the pattern that irregular quadrilaterals can tile a surface) and how these difficulties can be addressed in the classroom (cf. FAG 2). The box at the end of Figure 2 includes some further guidance about engaging students in tiling with an irregular quadrilateral.

The problems in Figures 3, 4, and 5 (examples 2, 3, and 4, respectively) were coded as proof tasks in which the support to teachers is ‘solution only.’ In example 2, the teacher’s edition provides a sketch of a solution to the task that points to a proof for Ji Young’s conjecture about divisibility by three of consecutive whole numbers. Although this solution could be unpacked to make the argument clearer, we can still say that teachers are provided with sufficient guidance about how to solve the task. Nevertheless, there is no guidance to teachers about other issues involved in solving the task. Given the robust findings of research that many students rely (incorrectly) on the use of empirical arguments to prove general mathematical statements (e.g., Chazan, 1993; Healy & Hoyles, 2000; Sowder & Harel, 1998), it is likely that many students in a class that uses CMP will use examples to show that Ji Young’s conjecture is true. Also, given the equally robust findings of research that many teachers, similarly to students, use empirical arguments to show the truth of general mathematical statements (e.g., Knuth, 2002; Martin & Harel, 1989; Selden & Selden, 2003; Simon & Blume, 1996), it is likely that teachers will accept students’ empirical arguments as appropriate ways to prove Ji Young’s conjecture. An example of an empirical argument for the conjecture would be the following:

> I’ve checked Ji Young’s conjecture for 18 sets of three consecutive whole numbers. In each set there’s always one number that is divisible by 3. For example, in the set {1, 2, 3} number 3 is divisible by 3. In the set {9, 10, 11} number 9 is divisible by 3. I’ve made sure to also check sets of large numbers. For example, I’ve checked the set {101, 102, 103} and it works! In this set, number 102 is divisible by 3. Therefore, the conjecture is always true.

Teachers could be further supported in implementing this task in their classrooms if the teacher’s edition: (1) cautioned teachers about the possibility of many students providing empirical arguments for the conjecture and offered an example of a possible student argument like the one above (cf. FAG 2); and (2) helped teachers understand how

Example 2 (from CMP unit, *Prime Time*)
Problem (p. 32): Ji Young conjectured that, in every three consecutive whole numbers, one number will be divisible by 3. Do you think Ji Young is correct? Explain.
Commentary in the Teacher’s Edition about this Problem (p. 33): Ji Young is correct; every third number is a multiple of 3, so every three consecutive numbers will contain a multiple of 3.

Figure 3. Guidance offered to teachers in proof tasks: example 2

Example 3 (from CMP unit, *Stretching and Shrinking*)
Problem (p. 38): Rectangle A is similar to rectangle B and also similar to rectangle C. Can you conclude that rectangle B is similar to rectangle C? Explain your answer. Use drawings and examples to illustrate your answer.
Commentary in the Teacher's Edition about this Problem (p. 40j): Yes, rectangles B and C are similar. Possible explanation: Since rectangle A is similar to rectangle B, the ratio of the short side of rectangle A to the long side of rectangle A is the same as the ratio of the short side of rectangle B to the long side of rectangle B. But since rectangle B is similar to rectangle C, the short side of rectangle C to the long side of rectangle C must equal this same ratio. This means the ratio between sides in rectangle C equals the ratio between sides in rectangle A, making rectangles C and A similar.

Figure 4. Guidance offered to teachers in proof tasks: example 3

empirical arguments differ from the expected proof (cf. FAG 3). I am not suggesting that if these two issues were discussed in the teacher's edition then the task would be enacted in the way intended by the curriculum authors. Nonetheless, such a discussion would provide one form of support that, at least, could make some teachers think more deeply about the issues involved.

In example 3, the teacher's edition provides a proof for the mathematical statement in the task about similarity of rectangles. As in example 2, the teacher's edition provides information to teachers about how to solve the task, but it does not offer teachers any additional guidance about issues of proof involved in the task. One way to further support teachers would be for the teacher's edition to elaborate on the last prompt of the task: "Use drawings and examples to illustrate your answer" (*Stretching and Shrinking*, p. 38). Given that both students and teachers are content to operating empirically, they may think (erroneously) that an *illustration* of their answer constitutes also a *justification* for it. (cf. FAG 2 and FAG 3)

In the proposed solution in example 4, the teacher's edition offers a *counterexample* to prove that the statement 'Subtraction of real numbers is a commutative operation' is false, but it does not elaborate on why this is a valid method for refuting the statement. Such an elaboration is crucial because research shows that both students and teachers have difficulty understanding the notion of counterexample. Silver & Carpenter (1989) report that in the 1986 National Assessment of Educational Progress in the USA, fewer than 50% of the 11th graders in the sample could select an appropriate counterexample. In a study with 12 to 15-year-old students, Galbraith (1981) found that over a third of the students did not understand that counterexamples satisfy the

Example 4 (from CMP unit, *Kaleidoscopes, Hubcaps, and Mirrors*)
Problem (p. 65): Is subtraction of real numbers a commutative operation?
Commentary in the Teacher's Edition about this Problem (p. 65): No; $5-3\neq3-5$.

Figure 5. Guidance offered to teachers in proof tasks: example 4

conditions of a conjecture but violate the conclusion. Most of the middle school students in Balacheff's (1988) experiment considered counterexamples as particular cases; when students' hypotheses were confronted with counterexamples they were unwilling to abandon their conjectures. Research with teachers reveals similar problems. A common misconception among teachers is that one counterexample does not suffice to prove false a general mathematical statement (Goetting, 1995; Simon & Blume, 1996; Sowder & Harel, 1998).

The findings of the research summarized above point to some additional forms of guidance that the teacher's edition could offer teachers in the context of example 4. For instance, it could explain to teachers that this task aims to engage students in a general mathematical statement: 'Subtraction of real numbers is a commutative operation' (cf. FAG 1). So, if one thinks that the statement is true, one needs to show that it is true for all possible cases. However, if one thinks that the statement is false, one only needs to find a counterexample. To emphasize further these ideas, the teacher's edition could contrast the proof by counterexample for the above statement with an empirical argument for a related true statement, such as 'Addition of real numbers is a commutative operation.' Then the teacher's edition could discuss the issue of the admissibility of proof by counterexample versus the inadmissibility of empirical arguments. (cf. FAG 3)

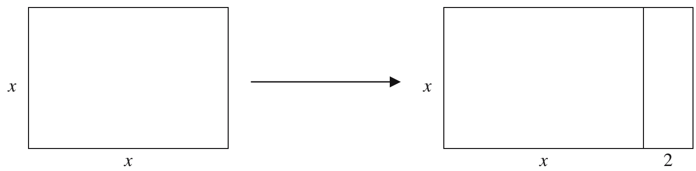
Set of Problems on the Topic of Equivalence of Symbolic Expressions

In this part, I discuss the guidance offered to teachers in six problems about the *equivalence of symbolic expressions*. As I explained earlier, an important reason for selecting this set of problems is that it offers the opportunity to discuss broader issues of curriculum design that would be difficult to discuss in the context of individual unrelated tasks. The six problems come from two consecutive eighth-grade CMP units. One goal in these units is for students to develop their ability to show the equivalence of symbolic expressions in different ways: by using tables, graphs, and diagrams; by using the underlying logic of the situation described in the problem; by applying properties of arithmetic. The order in which I discuss the six problems is the order the curriculum suggests that teachers follow when they implement the problems in their classrooms.[4]

In the two problems in Figure 6 (examples 5 and 6), the teacher's edition suggests the use of graphs for showing the equivalence of the given equations. The commentaries in the teacher's edition for the two problems not only fail to explain that drawing and comparing graphs of

Example 5 (from CMP unit, *Frogs, Fleas, and Painted Cubes*)

Problem (p. 22): A square has sides of lengths x centimeters. A new rectangle is created by increasing one dimension of the square by 2 centimeters.



1. The new rectangle is made up of the original square and an added rectangle. What are the dimensions of the added rectangle? What is its area?
2. Write an equation for the area of the new rectangle as the sum of the area of the original square and the area of the added rectangle.
3. What are the length and the width of the new rectangle? Write an equation for the area of the new rectangle as its length times its width.
4. Graph your equations from parts 2 and 3 on your calculator and copy the graphs onto your paper. Describe the shapes of the graphs. How do the graphs compare? What does this tell you about the two equations?

Commentary in the Teacher's Edition about Part 4 of this Problem (p. 40t): The graphs are identical parabolas with a minimum point at (-1, -1), x-intercepts at (0, 0) and (-2, 0), and symmetry about a vertical line through the minimum point. (Note: Students may observe only that the graphs are identical.) This means that the two equations are equivalent. [The graphs are also drawn in the teacher's edition.]

Example 6 (from CMP unit, *Frogs, Fleas, and Painted Cubes*)

Problem (p. 65): A jeweler has tried to increase her profit by lowering the price of her necklaces. Using past sale data, she has generated two equations relating income, i, to selling price, p:

$$i = (100 - p)p \text{ and } i = 100p - p^2$$

Are the two equations equivalent? How do you know?

Commentary in the Teacher's Edition about this Problem (p. 65): The equations are equivalent. Possible explanation: When you graph the equations, the graphs are identical, so the equations are the same.

Figure 6. Equivalence of symbolic expressions: examples 5 and 6

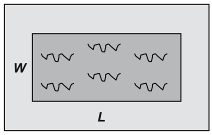
functions with an infinite domain can only suggest equivalence, but explicitly state that they show equivalence: “This means that the two equations *are* equivalent” and “The equations *are* equivalent” (*Frogs, Fleas, and Painted Cubes*, pp. 40t and 65, respectively; emphasis added). Despite the fact that the students are not expected to know until later in the curriculum about how to apply properties of arithmetic to show the equivalence of equations, there are mathematically accurate ways that students could employ to show the equivalence of the expressions in these two examples. In example 5, for instance, students could use the underlying logic of the situation described in the problem and say something like: ‘The two equations are equivalent because they represent the same quantity, namely the area of the new rectangle.’

The next four examples come from the subsequent CMP unit. Part c of the problem in Figure 7 (example 7) is the most relevant for my discussion here, because in this part students are asked to explain why the expressions from parts a and b are equivalent. A number of different possible responses are provided in the teacher’s edition for part c without,

Example 7 (from CMP unit, *Say It with Symbols*)

Problem (p. 26): a. How many 1-foot square tiles are needed to form a border for a pool that is 10 feet long and 5 feet wide?

b. Write an expression for the number of border tiles needed for a pool that is L feet long and W feet wide.



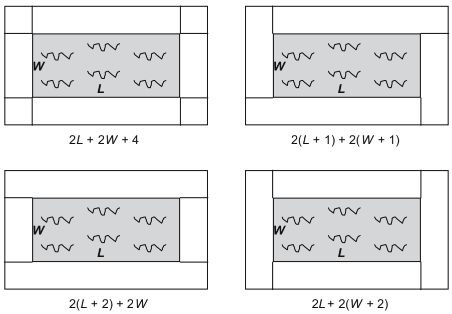
c. Write a different expression for the number of border tiles needed. Explain why your expressions are equivalent.

Commentary: The teacher's edition discusses this problem at two different places. The first provides possible responses to the questions in the problem, and the second relates to how to teach the problem to students. Next, I present the two commentaries as they appear in the teacher's edition.

Teacher's Edition p. 33l: a. $2(10) + 2(5) + 4 = 34$ tiles.

b. Possible expressions: $2L + 2W + 4$, $2(L + 1) + 2(W + 1)$, $2(L + 2) + 2W$, $2L + 2(W + 2)$.

c. See part b for some expressions; explanations will vary. Students might draw sketches; for example:



They might substitute values for L and W in the expressions; for example, when $W = 2$ and $L = 3$:
$2L + 2W + 4 = 2(3) + 2(2) + 4 = 14$, $2(L + 1) + 2(W + 1) = 2(4) + 2(3) = 14$,
$2(L + 2) + 2W = 2(5) + 2(2) = 14$, $2L + 2(W + 2) = 2(2) + 2(4) = 14$.

Students might also generate tables for their expressions. (Note: Generating tables for all possible values is impossible, so we can't say with certainty that the expressions are equivalent using this method. However, for students at this stage, identical tables indicate that the expressions are equivalent. Since there are two independent variables, making graphs would be difficult.)

Teacher's Edition p. 33d: [In this problem] students investigate a tile border around a rectangular pool. Let students work on the problem in pairs and share their answers with the class. Be sure they connect each part of the symbolic expression with their conceptualization of the situation and that they have attempted to explain why two different expressions are equivalent. (In this problem, using graphs or tables is problematic because of the number of variables.) Once several expressions have been produced, ask:

Use one of these expressions to find the number of tiles in the border of a pool 37 feet long and 19 feet wide. ...

This will encourage students to generalize. They should all find the same number of tiles. When students have had time to think about this question, ask them to write their own definitions for equivalent expressions.

Figure 7. Equivalence of symbolic expressions: example 7

however, these responses being organized in terms of their mathematical appropriateness. The sketches that are presented first in the teacher's edition could constitute appropriate geometric proofs if they were accompanied with a relevant commentary about how these sketches show the equivalence of the expressions. The possible student solution that appears after the sketches in the teacher's edition (about substituting specific values for L and W) constitutes an empirical argument. One could argue that teachers

could infer the limitations of this argument from the commentary that comes right after in the teacher's edition: "Generating tables for all possible values is impossible, so we can't say with certainty that the expressions are equivalent using this method" (*Say It with Symbols*, p. 26). Specifically, it is possible that teachers could make the connection between this commentary and the empirical argument that preceded it and think along the following lines: 'Substituting specific values is similar to generating tables: both methods are based on empirical explorations of a proper subset of all the possible cases. Therefore, checking for $W=2$ and $L=3$ cannot guarantee equivalence.' However, how likely is it that teachers (especially those who have difficulties with proof) will make these connections under the pressure of daily instruction? (cf. FAG 2)

Another part of the problem in example 7 that is worth attention is the following note to the teacher: "However, for students at this stage, identical tables indicate that the expressions are equivalent" (*Say It with Symbols*, p. 33l). It is unclear what the authors mean by the phrase 'at this stage.' This phrase cannot refer to students' developmental-psychological stage, because in examples 8 and 9 that I discuss next and that are taken from the *same* CMP unit, students are expected to understand the limitations of showing the equivalence of expressions by substituting specific values or by comparing tables and graphs. Thus, it seems that the phrase 'at this stage' refers to students' mathematical experiences up to this point. This interpretation is consistent with students' previous experiences with equivalent expressions as discussed in examples 5 and 6. The phrase also seems to be hinting to teachers that the approach of having students use tables to deduce equivalence will change in the curriculum. Yet, this information is given to teachers in an implicit way and does not indicate to them how the approach will change. (cf. FAG 1)

In the two problems in Figure 8 (examples 8 and 9), the teacher's edition becomes explicit on the issue of using graphs and tables to show the equivalence of expressions. Overall, the support it provides to teachers in these two examples is rich and well designed. The teacher's edition not only provides guidance to teachers on how to solve the tasks, but also points to possible student misconceptions (cf. FAG 2), helps teachers develop their own understanding of proof (cf. FAG 3), and prepares teachers to deal with issues that may come up when they implement the tasks in their classrooms (cf. FAG 2). Specifically, the teacher's edition explains to teachers that, "when we make tables and graphs for such relationships [functions with an infinite domain], it is impossible to list all the pairs of points or to draw the entire graph. We

<u>Example 8</u> (from CMP unit, *Say It with Symbols*)

Background: In the previous problem in the textbook (not presented here), the students discuss five expressions about the number of square tiles needed to construct a border around a square pool:

$$4s + 4,\ 4(s + 1),\ s + s + s + s + 4,\ 2s + 2(s + 2),\ 4(s + 2) - 4.$$

Example 8 is based on these five expressions.

Problem (p. 23): Evaluate each of the five expressions given in the problem for $s = 10$. Can you conclude from your results that all the expressions are equivalent? Explain your reasoning.

Commentary: The teacher's edition discusses this problem at two points. The first provides an answer to the problem and the second relates to how to teach the problem to students. Next, I present the two commentaries as they appear in the teacher's edition.

Teacher's Edition p. 33l: The five expressions yield the same result for this value of s. It is possible that the expressions are all equivalent, but we can't conclude this from checking one value.

$4s + 4 = 4(10) + 4 = 44$
$4(s + 1) = 4(10 + 1) = 44$
$s + s + s + s + 4 = 10 + 10 + 10 + 10 + 4 = 44$
$2s + 2(s + 2) = 2(10) + 2(10 + 2) = 44$
$4(s + 2) - 4 = 4(10 + 2) - 4 = 44$

Teacher's Edition pp. 33f-33g: [This problem] asks students to evaluate each expression for $s = 10$. They should find the same result for each expression.

Is one example enough to prove that the expressions are equivalent? (*One example or point on a graph is not enough, as there is an infinite number of solutions.*)

If students say yes, one example is enough, you may want to write these equations on the board and ask the questions that follow: $y = 2x + 1$ $y = x + 2$

Find y when $x = 1$. (*In each equation,* $y = 3$.)
Find y when $x = 3$. (*In the first equation,* $y = 7$*; in the second,* $y = 5$.)
Would it have been a good idea to conclude from our substitution that the expressions were equivalent? (*no*)

You might quickly sketch the graphs of the two equations and show that they intersect at the point (1, 3). This point lies on both graphs and is thus a solution to both equations. [The graphs are also drawn in the teacher's edition.]

<u>Example 9</u> (from CMP unit, *Say It with Symbols*)

Background: Same background as in Example 8.

Problem (p. 23): Make a table and a graph for each of the five expressions. Do the tables and the graphs indicate that the expressions are equivalent? Explain.

Commentary: The teacher's edition discusses this problem at two points. The first provides an answer to the problem and the second relates to how to teach it. Next, I present the two commentaries as they appear in the teacher's edition.

Teacher's Edition p. 33l: The graphs and tables of the expressions are the same, which supports the idea that the expressions are equivalent.

Teacher's Edition p. 33g: Help the class understand that when we make tables and graphs for such relationships, it is impossible to list all the pairs of points or to draw the entire graph. We can't prove equivalence by comparing these tables and graphs, but we can look at several values and be reasonably certain of it. Students might discuss whether the parts of the graphs and tables that are showing are sufficient to suggest equivalence.

Figure 8. Equivalence of symbolic expressions: examples 8 and 9

can't prove equivalence by comparing these tables and graphs, but we can look at several values and be reasonably certain of it." (*Say It with Symbols*, p. 33g) The teacher's edition also provides an educative (for the teachers) fictitious conversation in a class where some students argue that one example is enough to prove that the expressions in example 8 are equivalent. In particular, the teacher's edition provides a counterexample to the general statement that, 'If a point satisfies two linear equations then the equations are equivalent,' together with two different ways in which teachers can present this counterexample to their

students. Moreover, the language used in the teacher's edition in example 9 is accurate mathematically, contrary to the language used in examples 5 and 6 (which I discussed earlier) and the language used in example 10 (which I discuss next).

The problem in Figure 9 (example 10) does not seem to cohere with the thread of activities up to this point. Prior to example 7, the ways in which the tasks were designed and the guidance to teachers was presented were suggesting to students and teachers that tables and graphs offer valid methods to show equivalence. In example 7, teachers were provided with implicit guidance about the constraints of these methods. In examples 8 and 9, teachers and students were guided in a deliberate and thoughtful way to understand that these methods cannot be used to prove the equivalence of functions that have an infinite domain. Thus, up to example 9, there was a steady progression of explicitness in regard to what constitutes a valid method to show the equivalence of symbolic expressions. However, the phrasing of part b of the problem in example 10 contradicts what was promoted two pages earlier in the curriculum in examples 8 and 9. Specifically, part b in example 10 asks students to "[u]se tables and graphs to *show* that $5x+10$ and $5(x+2)$ are equivalent" (*Say It with Symbols*, p. 25; emphasis added). Instead of using the word 'show,' it would be more appropriate to use the phrase 'provide supporting evidence.'

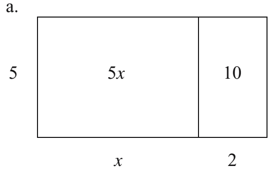
Example 10 (from CMP unit, *Say It with Symbols*)

Problem (p. 25): a. Draw a diagram that uses areas of rectangles to illustrate that $5x + 10$ and $5(x + 2)$ are equivalent.

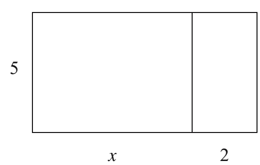
b. Use tables or graphs to show that $5x + 10$ and $5(x + 2)$ are equivalent.

Commentary in the Teacher's Edition about this Problem (p. 33m):

a.



area = $5x + 10$



area = $5(x + 2)$

b. Possible table:

x	$5x + 10$	$5(x + 2)$
1	$5 + 10 = 15$	$5(3) = 15$
6	$30 + 10 = 40$	$5(8) = 40$
13.5	$67.5 + 10 = 77.5$	$5(15.5) = 77.5$
20	$100 + 10 = 110$	$5(22) = 110$

The graphs of the equations $y = 5x + 10$ and $y = 5(x + 2)$ are the same. [The graphs are also drawn in the teacher's edition.]

Figure 9. Equivalence of symbolic expressions: example 10

The discussion of the six examples about the equivalence of symbolic expressions raises several interrelated issues that are central to the guidance provided to teachers, in the curriculum materials they use, to engage their students in proof tasks. Below I discuss three such issues that can have implications for curriculum design beyond CMP and the particular domain of proof.

The first issue has to do with the use of language. An inaccurate use of mathematical language might affect the learning of a classroom community and of students as future learners of mathematics (Ball & Bass, 2003). The inaccuracies found in the examples analyzed (cf. the discussion of examples 5, 6, and 10) may have negative consequences on both students' and teachers' understanding of what constitutes a mathematically legitimate way to show the equivalence of symbolic expressions. (cf. FAG 3)

The second issue is about achieving coherence among the different opportunities designed in the curriculum. In my earlier discussion, I concluded that, up to example 7, the set up of the tasks and the commentary in the teacher's edition communicate the message that it is acceptable to use tables and graphs to show the equivalence of equations with an infinite domain. Examples 8 and 9, however, help students and teachers understand that this is not true. Thus, there is incoherence in how the curriculum develops the topic of showing the equivalence of equations. A critical question here is how the incoherence could be managed so that the instructional sequence would be more meaningful for students and teachers. This question raises a broader challenge that curriculum authors face when they write curriculum materials, namely, how to reconcile two important and often competing considerations: students' developmental trajectory (e.g., 'Can students be introduced from the beginning of the unit to "the correct" way of proving equivalence of equations with an infinite domain?') and mathematical integrity (e.g., 'Does generating graphs constitute a valid way of showing the equivalence of equations with an infinite domain?') (for discussions on these considerations, see Ball, 1993; Dewey, 1903; Stylianides, 2007).

Curriculum materials have responsibility to not cultivate teacher or student misconceptions in any topic they design activities (cf. FAG 1). As Dewey (1903) noted, whatever the preliminary approach to learning is, it should not inculcate "mental habits and preconceptions which have later on to be bodily displaced or rooted up in order to secure proper comprehension of the subject" (p. 217). Martin & Harel (1989) express a similar idea in the domain of proof: "If teachers lead their students to believe that a few well-chosen examples constitute a proof, it is natural

to expect that the idea of proof in high school geometry and other courses will be difficult for the students" (pp. 41–42). This statement is equally applicable in the context of showing the equivalence of expressions: If teachers lead their students to believe that the examination of a few examples in a table or the comparison of graphs constitute proofs for the equivalence of expressions with an infinite domain, it is natural to expect that students' learning of proof will be hindered. For instance, it can become hard for students to accept the necessity of providing 'real' proofs when they examine the truth value of statements that involve an infinite number of cases. Finally, as I mentioned in my discussion of examples 5 and 6, there are mathematically accurate ways that students could use to show the equivalence of the expressions in these examples.

The third issue concerns the challenges that curriculum authors face in regard to the amount of, and the best way to present the, guidance to teachers in a teacher's edition. The optimization problem below captures some of these challenges: How can one write a teacher's edition that has 'manageable' length and from which teachers can benefit the most? For instance, there is a practical problem in repeating, in many different places in the teacher's edition, discussions like those in examples 8 and 9. One possible way to deal with this problem would be to refer teachers to specific places in the curriculum where they can find guidance about similar tasks. But still, there are no easy ways of managing the challenges that curriculum authors face. Also, the decisions made by curriculum authors in trying to manage these challenges may have different implications for different groups of teachers (Collopy, 2003; Davis & Krajcik, 2005; Remillard, 2000; Schneider & Krajcik, 2002). For instance, the CMP authors' decision to have the discussion about whether tables and graphs can prove equivalence later than earlier in the curriculum (cf. examples 8 and 9) is likely to favor teachers who have taught CMP before. Specifically, experienced teachers are likely to be aware of this discussion from the beginning of the unit and thus able to refer to it whenever they implement tasks on the topic of equivalence of expressions.

Conclusions

The curriculum is not a self-enacted entity; the teachers are the persons who will interpret the information the authors offer in the curriculum materials and make decisions about whether and how to implement curricular tasks

in their classroom. Therefore, even if there is a plethora of curricular tasks that are designed to engage students in particular topics, this does not mean that teachers will implement these tasks in the ways intended by the curriculum authors even if teachers would like to implement these tasks with high fidelity (Collopy, 2003; Corey & Gamoran, 2006; Remillard, 1999, 2005; Schneider & Krajcik, 2002; Stein et al., 1996). Yet, teachers would be better prepared to implement curricular tasks in topics that cause students difficulties and in which teachers themselves have a limited understanding if curriculum materials offer to them guidance that goes beyond possible solution(s) to tasks. This paper took a step toward building essential knowledge base for the development of curriculum materials that will provide teachers with necessary guidance to support their implementation of proof tasks. Next, I discuss implications for curriculum design and research, not only in the domain of proof but also more broadly, that derive from the analytic approach and empirical analysis contained herein.

The possible ways I discussed in which the guidance offered to teachers in particular proof tasks coded in the ‘solution only’ category could be further developed to qualify for the ‘solution with additional guidance’ category, can be useful for authors of different curriculum programs when they design guidance for teachers in the domain of proof. The analytic approach I used can serve as a prototype for curriculum analyses on other hard-to-teach and hard-to-learn topics in the school curriculum. For example, the process I followed to distinguish between proof tasks and non-proof tasks, using what I termed the ‘expected formulation’ of the tasks, would be equally applicable to a curriculum analysis focusing on problem solving in mathematics. Similarly, the three FAG I used in the examination of the guidance offered to teachers in the domain of proof would be equally applicable to examinations with a similar focus in topics in science such as developing explanations based on evidence, communicating and justifying findings, and making sense of the particulate model of matter (National Research Council, 2000). Curriculum analyses on the guidance teachers receive to implement tasks related to these (or other) hard-to-teach and hard-to-learn topics can benefit from the fact that such topics usually have attracted significant research attention and a lot is known about the difficulties that students and teachers face with them.

Further research is needed to examine whether and how the implementation of tasks in the domain of different hard-to-teach and hard-to-learn topics relates to the kind of guidance offered to teachers in the curriculum, and how the implementation of these tasks in the

classroom is mediated by different teacher characteristics such as teachers' content and pedagogical content knowledge, years of experience, years of implementing a certain curriculum program, whether or not the curriculum program aims to promote reform-oriented mathematics instruction. In the particular domain of proof, for example, it would be important to conduct parallel examinations of curriculum materials and classroom instruction that implements these materials to investigate questions like the following: Do teachers, who follow closely their curriculum materials, tend to implement proof tasks in the 'solution with additional guidance' category in different ways than they tend to implement proof tasks in the 'solution only' category, and, if so, how? How does the implementation of different proof tasks vary, if at all, with teachers' content and pedagogical content knowledge of proof? A more comprehensive analysis would also investigate how the issues of language and curriculum coherence, which surfaced in my analysis on the topic of equivalent expressions, might affect the implementation of certain curricular tasks. For example: Does an inaccurate use of language in a certain task affect the implementation of the task and student learning of proof that results from it, and, if so, how? How does this vary, if at all, with teachers' content and pedagogical content knowledge of proof?

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NOTES

[1] I use the phrase 'US reform-based curricula' to refer to curricula that were designed to align with the standards of the National Council of Teachers of Mathematics (NCTM, 1989, 2000). I use the phrase 'US conventional curricula' to refer to curricula that, for the most part, existed prior to the 1989 standards; while many of these curricula have been updated to address the NCTM standards, their updating is primarily one of retrofitting as opposed to complete re-design.

[2]I talk about opportunities *designed for* (as opposed to opportunities *offered to*) students in the curriculum materials to indicate my focus on the curriculum authors' intentions regarding these opportunities and to emphasize that a task's possibilities for engagement in proof may not be implemented as intended.

[3]In G. Stylianides (2005), I discuss issues and present findings related to how the same US reform-based mathematics curriculum designs opportunities for students, in the student's textbook, to engage in identifying patterns, making conjectures, providing arguments, and developing proofs.

[4]These six problems are not consecutive in the curriculum and are not the only problems in the two CMP units on the topic of equivalence of symbolic expressions. Nevertheless, the other problems on this topic do not change the essence of my discussion. For practical reasons, I focus on these six problems.

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